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Sr. No. of the Question Paper :

Name of the Course : CBCS B.Sc. (Honors) Mathematics

Unique Paper Code : 32351501_OC

Name of the Paper : C-11 (Metric Spaces)

Semester : V

Duration : 3 Hours

Maximum Marks : 75

Instruction for candidates

1. *Question No. 1 is compulsory. Attempt any 3 out of the remaining 5 questions.*
2. *All questions carry equal marks (18.75).*
3. *In the question paper, given notations have their usual meaning unless until stated otherwise. Also, if not mentioned specifically, $\mathbb{R}, \mathbb{R}^2, I_2$ or, any subset of \mathbb{R} etc. will be assumed to be endowed with the usual/standard metric.*

1 (a) State whether True or False. Justify your answer in brief.

(i) $d(x, y) = \sqrt{|x - y|}$ and $d^*(x, y) = (x - y)^2$ both define metrics on \mathbb{R} .

(ii) A function with discrete metric space as its domain is continuous.

(iii) $\mathbb{Q} \cup \{2^{1/2}\}$ is dense in \mathbb{R} .

(iv) Not every subset of a metric space is a metric subspace.

(v) A compact metric space is a complete metric space, but a complete metric space may not be compact metric space.

(vi) The set $[-1, 0] \cup \left\{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots\right\}$ is compact in \mathbb{R} .

2 × 6 = 12

(b) (i) Consider a metric space (X, d) and a 1-1 function $f : X \rightarrow \mathbb{R}$. Then verify whether the following defines a metric on X : $d^*(x, y) = d(x, y) + |f(x) - f(y)|$, $x, y \in X$. **3**

(ii) Show that the set $\{x \in \mathbb{Q} : x \in [0, 1]\}$ is closed in (\mathbb{Q}, d_u) .

1.75

- (iii) Let $A = \left\{ (x, y) \in \mathbb{R}^2 : y = \sin\left(\frac{1}{x}\right), 0 < x \leq 1 \right\}$. Write \bar{A} in (\mathbb{R}^2, d_u) . **2**
2. (a) (i) Prove that in a metric space (X, d) a sequence $\langle x_n \rangle$ is convergent and converges to a point x in X if and only if the sequence $\langle d(x_n, x) \rangle$ converges to 0 in \mathbb{R} . **2**
- (ii) Show that the sequence $\langle x_n \rangle$, where $x_n = \langle 1, 1/2, 1/3, \dots, 1/n, 0, 0, \dots \rangle$ converges to $x = \langle 1, 1/2, 1/3, \dots, 1/n, \dots \rangle$ in l_2 -space. **2**
- (iii) Write the boundary of the set $A = (0, 1) \cup \{2\}$ in \mathbb{R} . **1**
- (b) (i) Consider the subspace $X = I \cup J$ of \mathbb{R} , where $I =]0, 1[$ and $J = [4, 7[$. Show that $d(\sup_x I, I) > 0$. **2.75**
- (ii) Show that every non-empty set is a bounded metric space. **2**
- (iii) Consider \mathbb{R} endowed with discrete metric. Show that every singleton in it is both open and closed ball of radius less than 1. **2**
- (c) (i) Show that $A = \{\langle x_n \rangle \mid x_n = 0 \text{ or } 1\}$ is a metric subspace of l_∞ -space. What is the induced metric on A ? **3**
- (ii) Give an example of a sequence in the l_∞ -space which is co-ordinate wise convergent but not convergent in l_∞ . **4**
3. (a) Let (X_1, d_1) and (X_2, d_2) be metric spaces. Let $f: X_1 \rightarrow X_2$ be a continuous function. Define a metric d on X_1 as $d(a, b) = d_1(a, b) + d_2(f(a), f(b))$. Show that d is equivalent to d_1 . Moreover, prove that $f: (X_1, d_1) \rightarrow (X_2, d_2)$ is uniformly continuous. **4+2**
- (b) Let $X = C[0, 1]$. Consider the metrics d and ρ defined on $C[0, 1]$ by

$$d(f, g) = \sup \{|f(x) - g(x)| : x \in [0, 1]\};$$

$$\rho(f, g) = \int_0^1 |f(x) - g(x)| dx$$
 Is the identity map $i: (X, d) \rightarrow (X, \rho)$ a homeomorphism? Justify your answer. **5**
- (c) Suppose $f(x, y)$ is a continuous real valued function of the two real variables x and y . Prove that the curve in the Euclidean plane whose equation is $f(x, y) = 0$ is a closed set. Is $A = \{(x, y) \in \mathbb{R}^2 : y = x^2\}$ an open set? Justify your answer. **2+1**
- (d) Is (\mathbb{R}^2, d) isometric to (\mathbb{C}, e) , where d and e are standard metrics. Justify your answer. **3**
- (e) Let $A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \subset \mathbb{R}$. Find $d(\bar{A})$ when (i) d is usual metric on \mathbb{R} (ii) d is discrete metric on \mathbb{R} . **1.75**
4. (a) Let (X, d) be any metric space. Does there exist a bounded metric on X which is equivalent to d ? Justify your answer. **1+3**
- (b) Suppose f and g be real-valued functions on a metric space (X, d) . Let $A = \{x \in X : f(x) < g(x)\}$ and $B = \{x \in X : f(x) \neq 0\}$

Is A open? Is B open? Justify your answer. 3+2

(c) Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined as $f(x) = 1/(1+x^2)$. Is f continuous? Suppose $A \subseteq \mathbb{R}$ is a compact subset of \mathbb{R} . Is $f^{-1}(A)$ also compact? Give justification for your answer. 1+2

(d) Let $X = [1, 5] \cup [9, 12]$ endowed with usual metric inherited from \mathbb{R} . Is $\overline{S(10,5)} = \overline{S}(10,5)$? Justify your answer. 3

(e) Define $f: [0,1] \cup [2,3] \rightarrow [0,2]$ by

$$f(x) = \begin{cases} x & , x \in [0, 1) \\ 4-x & , x \in [2, 3] \end{cases}$$

Is f continuous? Is f bijective? Is f^{-1} continuous? Justify your answer. 1+1+1.75

5. Give suitable justification to your answer.

(a) Give an example of each of the following: A function defined as $f: \mathbb{R} \rightarrow \mathbb{R}$ having (i) no fixed point; (ii) having only 1 fixed point. 2

(b) Is $f: ([0,1], d_u) \rightarrow ([0,1], d_u)$ defined by $f(x) = x^2$ a contraction map? 2

(c) Give an example of an open set in \mathbb{R} which is complete but not connected. 2

(d) Give an example of a metric space where singletons are the only connected sets. 2

(e) Show that in \mathbb{R} , the continuous image of an interval is an interval. 3

(f) Can we have a countably infinite connected set in \mathbb{R} ? 2

(g) Can we have an uncountable set in \mathbb{R} which is not connected? 2

(h) Let $f: (X, d_X) \rightarrow (Y, d_Y)$ be a continuous, 1-1 and onto map, where (X, d_X) is a compact metric space. Let $A \subseteq Y$ be connected. Then what can be said about the connectedness of $f^{-1}(A)$? 2

(i) Comment upon the connectedness of the set $S = \{(x, y) \in \mathbb{R}^2 : xy = 1\}$. 1.75

6. Give suitable justification to your answer.

(a) Let $X = \mathbb{R} \setminus \{0\}$. Show that there exists a continuous map f defined on X such that

$$f(x) = \begin{cases} 1, & x \in (-\infty, 0) \\ 0, & x \in (0, \infty). \end{cases} \quad 4$$

(b) Can the closure of a connected set be disconnected? 2

- (c) Give an example of a metric space whose diameter is finite, but it is not compact. **2**
- (d) Give an example of a countable infinite set which is not compact, but its closure is compact. **2**
- (e) Give an example to justify that the continuous image of a bounded set need not be compact. **2**
- (f) Give an example of a collection of closed sets whose union is closed but not compact. **2**
- (g) Let $f : (X, d_X) \rightarrow (Y, d_Y)$ be a continuous functions and A be a closed and bounded subset of X . Then what can be said about the compactness of $f(A)$? **2**
- (h) An arbitrary intersection of compact sets is compact? **2.75**

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